



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 18

Duration & Immunization

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IF WE IGNORE CONVEXITY, THEN

**DURATION
AS LINEAR
APPROX**

$$\left. \frac{dP}{P} \right|_{y_0} = \frac{P'(y_0)}{P(y_0)} dy = -D_{\text{Mac}} \frac{dy}{(1+y_0)} \quad \text{or} \quad \left. \frac{dP}{dy} \right|_{y_0} = -D_{\text{Mac}} \frac{P(y_0)}{(1+y_0)}$$

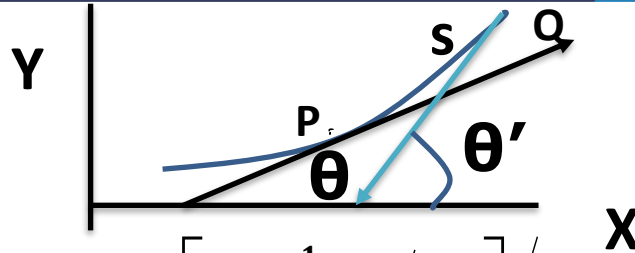
$$\text{where } D_{\text{Mac}} = -\frac{(1+y_0)}{P(y_0)} P'(y_0) = \frac{\sum_{t=1}^T \frac{tC_t}{(1+y_0)^t}}{P(y_0)} \text{ is fixed for given } y_0;$$

$$\text{or } \left. \frac{dP}{dy} \right|_{y_0} = P'(y_0) = \text{constant at given } y_0 \text{ showing that "duration" is a linear}$$

approximation of the yield-price curve around the point of reference.



CURVATURE



Curvature measures the rate at which the tangent line turns as we move along the curve.

$$\kappa = \lim_{\delta s \rightarrow 0} \frac{\delta \theta}{\delta s} = \frac{d\theta}{ds} = \frac{d\theta/dx}{ds/dx} = \frac{d[\tan^{-1} dy/dx]/dx}{ds/dx}$$

$$\frac{d}{dx} [\tan^{-1} (dy/dx)] = \frac{1}{1 + \left(\frac{dy}{dx}\right)^2} \frac{d^2 y}{dx^2}$$

$$\frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{1/2}$$

$$\kappa = \left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{-3/2} \frac{d^2 y}{dx^2}$$

Since (dy/dx) is simply the slope of the straight line approximation at the point under reference, it is the second derivative that captures the curvature effect.

IMPORTANT INFERENCE

- Higher the curvature (convexity) of a bond, greater will be the deviation of its actual price shift due to a shift in interest rates from the shift calculated using the “duration” formula.
- In other words, greater the convexity, higher will be the error by using the “only duration” approximation formula.



EXAMPLE 1

- Consider a 12% coupon bond with an yield to maturity of 18% and 5 years remaining to maturity.
 - a. What is the bonds current price, assuming annual coupons?
 - b. What is the bond's Duration? Convexity?
 - c. What percentage price change might you expect if the yield to maturity suddenly increased to 25%? Calculate using Duration alone and then using both Duration & Convexity.
 - d. What would be the exact percentage price change?



TIMELINE	0	1	2	3	4	5
YTM		0.18	0.18	0.18	0.18	0.18
DISC FACTOR		0.84745763	0.71818	0.60863	0.5157889	0.4371
CASH FLOW		12	12	12	12	112
DCF	<u>81.23697</u>	10.1694915	8.61821	7.30357	6.1894665	48.956
$tC(t)$		12	24	36	48	560
DISC $tC(t)$	<u>318.8557</u>	10.1694915	17.2364	21.9107	24.757866	244.78
DURATION	3.925007					
$t(t+1)C(t)$		24	72	144	240	3360
DISC $t(t+1)C(t)$	1752.167	20.3389831	51.7093	87.6428	123.78933	1468.7
CONVEXITY	10.7843					

ACTUAL REVISED PRICE						
TIMELINE	0	1	2	3	4	5
YTM		0.25	0.25	0.25	0.25	0.25
DISC FACTOR		0.8	0.64	0.512	0.4096	0.3277
CASH FLOW		12	12	12	12	112
DCF	<u>65.03936</u>	9.6	7.68	6.144	4.9152	36.7
ORIGINAL PRICE	81.23697					
ACTUAL % CHANGE	-0.199387					

GIVEN CHANGE IN YTM	0.07
% CHANGE IN PRICE USING DURATION	-0.232839
CONVEXITY CORRECTION	0.037951
NET % CHANGE IN PRICE	-0.194888

$$\frac{\Delta P}{P_0} = -D \left(\frac{dy}{1 + y_0} \right) + C \left(\frac{dy}{1 + y_0} \right)^2$$

$$\text{Hence, } \frac{\Delta P}{P_0} (\text{duration}) = -D \left(\frac{dy}{1 + y_0} \right) = -3.925 \left(\frac{0.07}{1 + 0.18} \right) = -0.2328$$

$$\text{Convexity correction} = C \left(\frac{dy}{1 + y_0} \right)^2 = 10.78 \left(\frac{0.07}{1 + 0.18} \right)^2 = 0.0379$$

Hence, net change = -19.49%

MODIFIED DURATION

$$\text{Duration } D_{\text{Mac}} = - \frac{(1 + y_0) P'(y_0)}{P(y_0)} = \frac{\sum_{t=1}^T \frac{tC_t}{(1 + y_0)^t}}{P(y_0)}$$

$$\text{Mod Duration} = \frac{D_{\text{Mac}}}{(1 + y_0)} = - \frac{P'(y_0)}{P(y_0)} \text{ so that}$$

$$\left. \frac{dP}{P} \right|_{y=y_0} = -D_{\text{Mod}} \times dy \text{ or } D_{\text{Mod}} = \left. \frac{dP}{P} \right|_{y=y_0} / dy$$

MODIFIED DURATION: DEFINITION

- The modified duration of a bond at a given YTM is the negative of percentage change in price of the bond for a 1% change in YTM.
- Since price & YTM are inversely related, modified duration returns a positive number for a conventional bond.



DURATION: SEMI-ANNUAL BONDS

- The duration of a semiannual-pay bond can be calculated in the same way:
- as a weighted average of the number of *semiannual periods* until the cash flows are to be received. In this case, the result is the number of semiannual periods rather than years.

- For a semiannual-pay bond with a YTM quoted on a semiannual bond basis:
- $D_{\text{Mod,Semi}} = D_{\text{Mac,Semi}} / (1 + y/2)$
- This modified duration can be annualized (from semiannual periods to annual periods) by dividing by two, and then used as the approximate percentage change in price for a 1% change in a bond's YTM.

INTEREST RATE ELASTICITY

$$\mathbf{IE}_{y_0} = \frac{dP/P_0}{dy/y_0} = \frac{P'(y_0)}{P(y_0)} y_0 = - \left(\frac{D_{\text{Mac}}}{1 + y_0} \right) y_0 = -D_{\text{Mod}} y_0$$

$$\text{since } P'(y_0) = -D_{\text{Mac}} \frac{P(y_0)}{(1 + y_0)} = -D_{\text{Mod}} \times P(y_0)$$

(IGNORING CONVEXITY)



RECAP

$$D_{\text{MOD}} = -\frac{P'(y_0)}{P(y_0)} = \frac{\frac{1}{(1+y_0)} \sum_{t=1}^T \frac{tC_t}{(1+y_0)^t}}{P(y_0)} = \frac{D_{\text{MAC}}}{(1+y_0)}$$

$$DV01 = -\frac{dP}{dy} = D_{\text{MOD}} \times P = \frac{D_{\text{MAC}}}{1+y} \times P$$

$$\left. \frac{\Delta P}{P} \right|_{y_0, P_0} = -D_{\text{MAC}} \frac{dy}{(1+y_0)} + C \left(\frac{dy}{1+y_0} \right)^2 = -D_{\text{Mod}} dy + C \left(\frac{dy}{1+y_0} \right)^2$$

$$IE = \frac{dP/P_0}{dy/y_0} = -D_{\text{MAC}} \left(\frac{y_0}{1+y_0} \right) = -D_{\text{Mod}} y_0$$



PROBLEM

- Consider a 10% annual coupon bond with a maturity of two years. The bond is quoting at a YTM of 18% p.a. Coupons will be paid at the end of the first and second year. Redemption at a discount of 25% to face value will be made at the end of second year. Calculate the bond's Macaulay's duration (in years) & convexity (in years²).

$$\begin{aligned}
 D &= \frac{\sum_{t=1}^T \frac{tC_t}{(1+y_0)^t}}{P(y_0)} = \frac{\sum_{t=1}^T \frac{tC_t}{(1+y_0)^t}}{\sum_{t=1}^T \frac{C_t}{(1+y_0)^t}} = \frac{\frac{1 \times 100}{1.18^1} + \frac{2 \times (100 + 750)}{1.18^2}}{\frac{100}{1.18^1} + \frac{(100 + 750)}{1.18^2}} \\
 &= \frac{84.75 + 1220.91}{84.75 + 610.46} = \frac{1305.66}{695.21} = 1.8781
 \end{aligned}$$

$$\begin{aligned}
 C &= \frac{\sum_{t=1}^T \frac{t(t+1)C_t}{(1+y_0)^t}}{2P(y_0)} = \frac{\frac{1 \times 2 \times 100}{1.18^1} + \frac{2 \times 3 \times (100 + 750)}{1.18^2}}{2 \times \left(\frac{100}{1.18^1} + \frac{(100 + 750)}{1.18^2} \right)} \\
 &= \frac{169.50 + 3662.73}{84.75 + 610.46} = \frac{3932.23}{2 \times 695.21} = 2.7562
 \end{aligned}$$

LET FACE VALUE BE	1000		
YEAR		1	2
COUPON PAYMENT	0.1	100	100
REDEMPTION VALUE	0.75		750
TOTAL CASH FLOWS		100	850
YTM	0.18	0.18	0.18
DISCOUNT RATE		0.847457627	0.7181844
CURRENT PRICE	695.20253	84.74576271	610.45677
TIME WEIGHTED CASHFLOWS	1305.6593	84.74576271	1220.9135
DURATION	1.8780992		
t(t+1) WEIGHTED CASHFLOWS	3832.2321	169.4915254	3662.7406
CONVEXITY	2.7561983		

PROBLEM

- On the basis of the value of duration & convexity arrived at in the previous question,, calculate the price change (in %) corresponding to a 5% increase in the ytm from the above figure.

GIVEN YTM (y)	0.18
CHANGE IN YTM (dy)	0.05
HENCE, $dy/(1+y)$	0.0423729
% CHANGE IN PRICE DUE TO DURATION	-0.0795805
% CORRECTION DUE TO CONVEXITY	0.0049486
TOTAL PRICE CHANGE	-0.0746318

ACTUAL PRICE CHANGE			
LET FACE VALUE BE	1000		
YEAR		1	2
COUPON PAYMENT	0.1	100	100
REDEMPTION VALUE	0.75		750
TOTAL CASH FLOWS		100	850
YTM	0.23	0.23	0.23
DISCOUNT RATE		0.81300813	0.66098222
CURRENT PRICE	643.1357	81.30081301	561.834887
% PRICE CHANGE	-0.0749		

IMMUNIZATION

Consider a case when the interest rates increase immediately after a bond investment.

Obviously, the income from reinvested coupons will increase due to this increase in interest (reinvestment) rates.

However, the anticipated price at the end of the holding period will decrease and hence, the expected capital gains would decrease.



Thus, the reinvestment income and the capital gains move in opposite directions due to a change in interest rate and hence, annul the effect of each other to some extent.



IMPACT OF HOLDING PERIOD ON BOND RETURNS

Nevertheless, the extent to which these effects would cancel each other also depends on the holding period of the investor.

The longer the holding period, the greater would be the effect on reinvested income of an interest rate change and smaller would be the effect on price since the bond would be closer to maturity and hence, fewer coupons would be available for discounting.



In fact, there exists a holding period at which both these effects exactly annul each other. This holding period turns out to be the Macaulay's duration as proved below:



We have $P_0 = \sum_{t=1}^T \frac{C_t}{(1+y)^t}$

Let H be the holding period of the investor

Then, total cash flows to the investor at $t = H$

$TCF(y, H) =$ proceeds of reinvested coupons + sale price

$$= \sum_{t=1}^H C_t (1+y)^{H-t} + P_H$$



$$\text{But } P_H = \sum_{j=1}^{T-H} \frac{C_{H+j}}{(1+y)^j} = \sum_{t=H+1}^T \frac{C_t}{(1+y)^{t-H}} = \sum_{t=H+1}^T C_t (1+y)^{H-t}$$

$$\text{Hence, } TCF(y, H) = \sum_{t=1}^H C_t (1+y)^{H-t} + P_H$$

$$= \sum_{t=1}^H C_t (1+y)^{H-t} + \sum_{t=H+1}^T C_t (1+y)^{H-t} = \sum_{t=1}^T C_t (1+y)^{H-t}$$

$$\text{TCF}(y, H) = \sum_{t=1}^T C_t (1+y)^{H-t}$$

$$\Rightarrow \frac{d\text{TCF}}{dy} = \sum_{t=1}^T (H-t) C_t (1+y)^{H-t-1} = (1+y)^{H-1} \sum_{t=1}^T (H-t) C_t (1+y)^{-t}$$

$$= (1+y)^{H-1} \left[H \sum_{t=1}^T C_t (1+y)^{-t} - \sum_{t=1}^T t C_t (1+y)^{-t} \right]$$

$$\text{Thus, } \frac{d\text{TCF}}{dy} = 0 \Rightarrow H(y) = \frac{\sum_{t=1}^T t C_t (1+y)^{-t}}{\sum_{t=1}^T C_t (1+y)^{-t}} = \frac{\sum_{t=1}^T t C_t (1+y)^{-t}}{P_0} = D_{\text{Mac}}$$

- Thus, given any rate (y), if an investor adopts a holding period $H(y)$ given by

$$H(y) = \frac{\sum_{t=1}^T tC_t (1+y)^{-t}}{\sum_{t=1}^T C_t (1+y)^{-t}}$$

- then his total cash flows from the investment are immunized against changes in the interest rate.

THE CAVEAT

- The immunization of the cash flows extends over an infinitesimally small range $y-dy$ to $y+dy$ of interest rate y .
- This is because Newtonian differentiation presupposes a straight line structure in the infinitesimal neighborhood of the point at which the curve is differentiated.



- Because, the price/yield curve is non-linear, the immunization will not extend over a large variation in y .
- Furthermore, it is also assumed that the value of dy is the same for all maturities i.e. the shift of the yield curve is parallel to itself.